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Irreducible affine space curves and the uniform Artin–Rees property on the prime spectrum

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Abstract

We give a negative answer to the problem, open for twenty years, as to whether the full uniform Artin–Rees property holds on the prime spectrum of an excellent ring (it was known to hold locally on the prime spectrum of such a ring).

Keywords: Uniform Artin–Rees property; Relation type; Herzog ideal; Northcott ideal

It is known that the full uniform Artin–Rees property holds locally on the prime spectrum of an excellent ring, and it has been a long-standing open problem whether this property continues to hold if the localisation is dispensed with (see below for details). The purpose of this paper is to give a negative answer to this question by proving the following result.

Theorem 1. *Let k be a field and $A = k[X_1, X_2, X_3]$ the polynomial ring in the variables X_1, X_2, X_3. Set $M = A$ and $N = X_3M \subset M$. For an integer $n \geqslant 4$, let $f_1(n) = n^2 - 3n + 1$, $f_2(n) = n^2 - 3n + 3$ and $f_3(n) = n^2 - n + 1$. Let $\mathfrak{p}_n \subset A$ be the vanishing ideal of the irreducible affine space curve of $\mathbb{A}^3_k$ given by the parametrisation $x_1 = t^{f_1(n)}$, $x_2 = t^{f_2(n)}$, $x_3 = t^{f_3(n)}$. Then*

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$$\mathfrak{p}_n^n M \cap N \supsetneq \mathfrak{p}_n\big(\mathfrak{p}_n^{n-1} M \cap N\big).$$

This result answers the question of Duncan and O'Carroll in [2, §2,(a)]. We begin by describing this question.

Let A be a noetherian ring, I an ideal of A and let $N \subseteq M$ be two finitely generated A-modules. The Artin–Rees lemma states that there exists an integer $s \geqslant 1$, depending on N, M and I, such that for all $n \geqslant s$,

$$I^n M \cap N = I^{n-s}\big(I^s M \cap N\big).$$

In particular, $I^n M \cap N \subseteq I^{n-s} N$ for such $n \geqslant s$.

Suppose we are given fixed A, M and N as before, together with a class $\mathcal{C}$ of ideals of A. If there exists an integer $s \geqslant 1$ such that the previous displayed equality holds for all $n \geqslant s$ and for all ideals $I \in \mathcal{C}$ then we say that the *full uniform Artin–Rees property* holds for the class $\mathcal{C}$.

Eisenbud and Hochster asked in [3] whether the full uniform Artin–Rees property held for the class of maximal ideals in an affine ring. In the same paper they gave a negative answer in the non-affine case. O'Carroll in [14] solved the "containment" version of the problem and proved essentially that if A is excellent, then there exists an integer s such that $\mathfrak{m}^n M \cap N \subseteq \mathfrak{m}^{n-s} N$, for all $n \geqslant s$ and for all maximal ideals $\mathfrak{m}$ of A. And soon after, Duncan and O'Carroll in [1] solved the full problem in excellent rings. They also remarked in [2] that in fact their proof establishes that there exist an integer $s \geqslant 1$ such that, for all $n \geqslant s$ and for all prime ideals $\mathfrak{p}$ of an excellent ring A,

$$\big(\mathfrak{p}^n M \cap N\big)_{\mathfrak{p}} = \big(\mathfrak{p}^{n-s}\big(\mathfrak{p}^s M \cap N\big)\big)_{\mathfrak{p}}. \qquad \text{(i)}$$

(See also the proofs of [18, Theorem 4] and [17, Theorem 5.2], from which one can deduce the same result.) They then asked in [2] whether the question of Eisenbud and Hochster holds for classes of ideals other than maximal ideals and, in particular, if one can drop in the equality (i) the localisation at the prime $\mathfrak{p}$.

In this direction, O'Carroll proved in [15] the validity of the full uniform Artin–Rees property for the class of principal ideals of a noetherian ring. This property is also true for the set of ideals having a reduction generated by a non-zerodivisor in a noetherian ring with finite integral closure [5] and for the whole set of ideals of an excellent ring provided that $\dim(M/N) \leqslant 1$ (see [16]). But if $\dim(M/N) = 2$, then one cannot expect the full property for the set of all ideals of A. Indeed, Wang in [21] showed that if $(A, \mathfrak{m})$ is a 3-dimensional regular local ring, $\mathfrak{m} = (x_1, x_2, x_3)$ and $J = (x_3)$, then there does not exist an integer $s \geqslant 1$ for which the equality $I^n A \cap J = I^{n-s}(I^s A \cap J)$ holds for all $n \geqslant s$, for all ideals I of the form $I_n = (x_1^n, x_2^n, x_1 x_2^{n-1} + x_3^n)$. With respect to the "containment" version of the problem, Huneke showed that there is an integer s for which the containment $I^n M \cap N \subseteq I^{n-s} N$ holds for all $n \geqslant s$ and for all ideals I of A, if A is a noetherian ring satisfying some extra conditions, e.g. if A is a noetherian ring which is essentially of finite type over a noetherian local ring (see [8] and [9]), and he conjectured that the result is true for any excellent noetherian ring of finite Krull dimension.

Theorem 1 says that the question of Duncan and O'Carroll does not hold for the prime spectrum of an excellent ring. In other words, one cannot drop in the equality (i) the localisation at the prime ideal $\mathfrak{p}$.

Call $s(N, M; I)$ the minimum of all integers $s \geqslant 1$ for which the equality $I^n M \cap N = I^{n-s}(I^s M \cap N)$ holds for all $n \geqslant s$. Remark that if $s(N, M; I) = s$, then necessarily $I^s M \cap N \supsetneq I(I^{s-1} M \cap N)$, because if not, for all $p \geqslant 1$,

$$I^{s+p} M \cap N = I^p\bigl(I^s M \cap N\bigr) = I^p I\bigl(I^{s-1} M \cap N\bigr) = I^{p+1}\bigl(I^{s-1} M \cap N\bigr)$$

and $s(N, M; I) \leqslant s - 1$. Following the idea of Wang's example, we are going to calculate the Artin–Rees number $s(N, M; I)$ by means of the relation type of a related ideal (see [21, Example 6.1 and Proposition 6.2]; see also [16, Theorem 2]). Let $\mathcal{R}(I; M) = \bigoplus_{n\geqslant 0} I^n M$ be the Rees module of I with respect to the module M. Recall that the relation type of $\mathcal{R}(I; M)$, denoted by $\mathrm{rt}(I; M)$, is the largest degree of a minimal set of relations of a presentation of $\mathcal{R}(I; M)$ as a quotient of a polynomial module with coefficients in M. The ideal I is said to be of linear type with respect to M if $\mathrm{rt}(I; M) = 1$. If $M = A$ one removes the "M" and the phrase "with respect to M." It is shown in [16, Theorem 2] that

$$s(N, M; I) \leqslant \mathrm{rt}(I; M/N) \leqslant \max\bigl(\mathrm{rt}(I; M), s(N, M; I)\bigr).$$

In particular, if $\mathrm{rt}(I; M) = 1$, then $s(N, M; I) = \mathrm{rt}(I; M/N)$.

So the first step to deal with the question of Duncan and O'Carroll is to find a large class of prime ideals of relation type 1. We consider the vanishing ideals $\mathfrak{p}$ of irreducible affine space curves given by parametric equations $x_1 = t^{n_1}$, $x_2 = t^{n_2}$, $x_3 = t^{n_3}$, with $\gcd(n_1, n_2, n_3) = 1$, i.e. the kernel of morphisms $\varphi : k[X_1, X_2, X_3] \to k[T]$ with $\varphi(X_i) = T^{n_i}$. It was proved by Herzog in [7] (see also [10, pages 138–139]) that $\mathfrak{p}$ is a prime ideal generated by either two or three elements. In the first case, the generators form a regular sequence, which implies that the ideal is of linear type (see [11]). If $\mathfrak{p}$ is generated by three elements, then $\mathfrak{p}$ is what Hermann, Moonen and Villamayor call an almost complete intersection. They proved in [6, Theorem 4.8], that if A is a Cohen–Macaulay ring, then an ideal which is an almost complete intersection is also of linear type. Thus, it remains to choose the adequate family of parametric curves with vanishing ideals $\mathfrak{p}_n \subset A = k[X_1, X_2, X_3]$ and an ideal J of the ring A such that the relation type $\mathrm{rt}(\mathfrak{p}_n; A/J)$ is unbounded. More specifically, we choose the $\mathfrak{p}_n$ in such a way that the $\mathfrak{p}_n A/J$ coincide with the ideals $I_n A/J$, where the I_n are the given ideals in the example of Wang.

Proof of Theorem 1. For an integer $n \geqslant 4$, let $f_1(n) = n^2 - 3n + 1$, $f_2(n) = n^2 - 3n + 3$ and $f_3(n) = n^2 - n + 1$. Remark that $\gcd(f_1(n), f_2(n)) = 1$ since $f_1(n)$ and $f_2(n)$ are odd numbers and $f_2(n) - f_1(n) = 2$. Let $\mathfrak{p}_n$ be the kernel of the k-homomorphism $\varphi_n : k[X_1, X_2, X_3] \to k[T]$ defined by $\varphi_n(X_i) = T^{f_i(n)}$. Let $\mathfrak{a}_n$ be the ideal of $A = k[X_1, X_2, X_3]$ generated by the three polynomials

$$\begin{aligned} F_{1,n} &= X_1^n - X_2 X_3^{n-3}, \\ F_{2,n} &= X_2^n - X_1^{n-1} X_3, \\ F_{3,n} &= X_3^{n-2} - X_1 X_2^{n-1}. \end{aligned}$$

Clearly $\mathfrak{a}_n \subseteq \mathfrak{p}_n$. Let us prove that $\mathfrak{a}_n$ is a height two prime ideal of A, so that $\mathfrak{a}_n = \mathfrak{p}_n$.

(1) *The elements $F_{1,n}, F_{2,n}$ form a regular sequence.* First X_3 is regular modulo $F_{1,n}A$. Moreover, $F_{1,n}A_{X_3}$ is a prime ideal as $A_{X_3}/F_{1,n}A_{X_3} \cong k[X_1, X_3, X_3^{-1}]$, since in A_{X_3}, $X_2 =$

$X_3^{-(n-3)}X_1^n$. Hence $F_{1,n}A$ is a prime ideal. Further, $F_{2,n} \notin F_{1,n}A$, as is clear on setting $X_2 = 0$ when we suppose the contrary.

(2) *The ideal* $\mathfrak{a}_n$ *is a Northcott ideal and is unmixed of height* 2 (see e.g. [20, p. 100] or [13]). Let $[\mathbf{u}_n] := [X_1^{n-1}, -X_2]$ and let ϕ_n be the 2×2 matrix defined by

$$\phi_n = \begin{pmatrix} X_1 & X_3^{n-3} \\ -X_3 & -X_2^{n-1} \end{pmatrix}.$$

Then $\phi_n \cdot [\mathbf{u}_n]^\top = [\mathbf{v}_n]^\top$ where $[\mathbf{v}_n] := [F_{1,n}, F_{2,n}]$. Since the grade of the ideal $(F_{1,n}, F_{2,n})$ is two and $\det(\phi_n) = F_{3,n}$, $\mathfrak{a}_n = (\mathbf{v}_n, \det(\phi_n)) = (\mathbf{v}_n) : (\mathbf{u}_n)$ (see [20, Corollary 4.1.1]). Since $(\mathbf{v}_n)$ is a complete intersection ideal of height 2, the rest of (2) follows from standard properties of the colon operation and primary ideals.

(3) *The ideal* $\mathfrak{a}_n$ *is prime*. Note first of all that X_1 is regular modulo $\mathfrak{a}_n$. For if $X_1 \in \mathfrak{p}$, for some $\mathfrak{p} \in \mathrm{Ass}_A(A/\mathfrak{a}_n)$, it easily follows that $\mathfrak{p} = (X_1, X_2, X_3)$ is of height 3. This contradicts (2).

Thus it suffices to show that $\mathfrak{a}_n A_{X_1}$ is a prime ideal. But $\mathfrak{a}_n A_{X_1} = (\mathbf{v}_n) A_{X_1}$ and in $A_{X_1}/(\mathbf{v}_n)A_{X_1}$, $X_3 = X_1^{-(n-1)} X_2^n$. Hence, by an easy calculation, on multiplying through by the unit $X_1^{(n-1)(n-3)}$,

$$A_{X_1}/(\mathbf{v}_n)A_{X_1} \cong k\big[X_1, X_2, X_1^{-1}\big]/\big(X_1^{f_2(n)} - X_2^{f_1(n)}\big).$$

Now note that in $k[X_1, X_2]$, $\{X_1^m\}_m \cap (X_1^{f_2(n)} - X_2^{f_1(n)}) = \emptyset$. Hence (3) follows from the fact that in $k[X_1, X_2]$, the ideal $(X_1^{f_2(n)} - X_2^{f_1(n)})$ is prime (see e.g. [4, Lemma 10.15]).

End of the proof. Since $\mathfrak{a}_n$ is a height two prime ideal of A, then $\mathfrak{a}_n = \mathfrak{p}_n$. Thus $\mathfrak{p}_n A/(X_3) = (X_1^n, X_2^n, X_1 X_2^{n-1})$. Clearly, $\mathrm{rt}(\mathfrak{p}_n A/(X_3)) = n$ (see e.g. [21, Proof of Example 6.1] or [16, Remark 4.3]). Since, as remarked just before the beginning of the proof, $\mathrm{rt}(\mathfrak{p}_n) = 1$, then $s((X_3), A; \mathfrak{p}_n) = \mathrm{rt}(\mathfrak{p}_n A/(X_3)) = n$ and $\mathfrak{p}_n^n A \cap (X_3) \supsetneq \mathfrak{p}_n(\mathfrak{p}_n^{n-1} A \cap (X_3))$. $\square$

Closing Remarks.

(I) The situation we deal with does not seem to be amenable to the approach used by Kunz [10] in the general case. We assume that, in turn, Kunz evolved his approach (a variant on Herzog's original arguments [7]) to cope with the fact that Northcott's argument for the particular case $n_1 = 3$, $n_2 = 4$, $n_3 = 5$ (cf. [12, p. 29]) does not extend in any obvious way to the general case.

(II) Note that we have shown that the "Herzog" ideal $\mathfrak{p}_n$ is a Northcott ideal. In fact we have the following general observation: *Every Herzog ideal is a Northcott ideal*. This follows as in (2) above, on employing Kunz's treatment of Herzog ideals and his notation (cf. [10, pp. 137ff and 139, line 15]), if we set

$$\phi = \begin{pmatrix} X_1^{r_{31}} & X_3^{r_{13}} \\ -X_3^{r_{23}} & -X_2^{r_{32}} \end{pmatrix}, \qquad \mathbf{u} = \big(X_1^{r_{21}}, -X_2^{r_{12}}\big).$$

(III) Recall that Herzog ideals can also be exhibited as a determinantal ideal of the form $I_2(\mathcal{M})$, where $\mathcal{M}$ is a 2×3-matrix whose entries are powers of the variables X_1, X_2, X_3 (cf. [19, Section 3]). Indeed M.E. Rossi pointed out to us that $\mathfrak{a}_n$ is equal to the ideal $I_2(\mathcal{M}_n)$ where $\mathcal{M}_n$ is the 2×3 matrix with rows $X_1^{n-1}, X_2^{n-1}, X_3^{n-3}$ and X_2, X_3, X_1 respectively.

(IV) S. Goto has subsequently outlined an alternative proof that $\mathfrak{a}_n = \mathfrak{p}_n$.

Tensoring the sequence $A/\mathfrak{a}_n \to A/\mathfrak{p}_n \to 0$ with A/X_1A and using a graded version of Nakayama's lemma, we see that it suffices to prove that the Artinian rings $A/(\mathfrak{a}_n + X_1A)$ and $A/(\mathfrak{p}_n + X_1A)$ have the same length. Since $\mathfrak{a}_n + X_1A$ is a monomial ideal, the first length is easily seen to be $f_1(n)$. Let $R = k[t^a, t^b, t^c]$, where for brevity we set $a = f_1(n), b = f_2(n)$ and $c = f_3(n)$, and let $S = k[t]$. We may harmlessly localise throughout at $A\backslash(X_1, X_2, X_3)$, which we do without changing notation. The second length is then $\lg(R/t^aR)$, which equals $e(t^aR)$, in the usual terminology of multiplicity theory. In turn, by [22, Corollary 1, p. 299], $e(t^aR) = e(t^aS)$, and $e(t^aS) = a = f_1(n)$, as required.

Afterwards, G. Valla presented an analogous proof centred on the fact that t^a is a superficial element in R; indeed t^a is a minimal reduction of $(t^a, t^b, t^c)R$.

(V) Since $A = k[X_1, X_2, X_3]$ is a noetherian ring essentially of finite type over a noetherian local ring, due to the aforementioned result of Huneke there does exist an integer $s \geqslant 1$ such that $\mathfrak{p}_n^m A \cap (X_3) \subseteq \mathfrak{p}_n^{m-s}(X_3)$ for all $m \geqslant s$ and for all $\mathfrak{p}_n$. Moreover, by the result of Duncan and O'Carroll mentioned before, there does exist an integer $t \geqslant 1$ such that $(\mathfrak{p}_n^m A \cap (X_3))_{\mathfrak{p}_n} = (\mathfrak{p}_n^{m-t}(\mathfrak{p}_n^t A \cap (X_3)))_{\mathfrak{p}_n}$ for all $m \geqslant t$ and for all $\mathfrak{p}_n$. On the other hand, with respect to the full property, one even has that

$$\mathfrak{p}_n^n A \cap (X_3) \supsetneq \mathfrak{p}_n\big(\mathfrak{p}_n^{n-1} A \cap (X_3)\big) + \mathfrak{m}\mathfrak{p}_n^n A \cap (X_3),$$

where $\mathfrak{m}$ is the maximal ideal $\mathfrak{m} = (X_1, X_2, X_3)$. This follows from the fact that $\mathrm{rt}(\mathfrak{p}_n A/(X_3))$ coincides with $\mathrm{rt}_{\mathfrak{m}}(\mathfrak{p}_n A/(X_3))$, the relation type of the fibre cone with respect to the maximal ideal $\mathfrak{m}$ (see [5] for more details).

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